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Functional Approach to Scattering in Quantum Field Theory

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A functional approach to scattering theory in quantum field theory is developed by deriving an explicit functional expression for *transition* amplitudes. In applications, the formalism avoids dealing with noncommutativity problems of field operators, avoids solving the field equations, avoids dealing with the often quite complicated continual (path) integrals, and avoids combinatoric problems associated with Feynman rules and the old-fashioned Wick's theorem. Finally, it avoids explicitly taking mass shell limits as in the LSZ formalism. The basic idea of the formalism is to use the quantum action principle followed by a systematic analysis of the concept of stimulated emissions as applied to particles of any spin, and is a generalization of an earlier method applied by the author to the much simpler situation of quantum mechanics.

1. INTRODUCTION

The purpose of this paper is to generalize the functional approach to scattering theory developed earlier (Manoukian, 1987) in quantum mechanics to quantum field theory. We derive an explicit functional expression for *transition* amplitudes for scattering in quantum field theory. The method of applications avoids dealing with noncommutativity properties of field operators usually encountered, avoids solving any field equations, and avoids dealing with the often quite complicated continual (cf. Feynman and Hibbs, 1965; Faddeev, 1981; Manoukian, 1985) path integrals, as it provides a solution to the latter. It also avoids dealing with combinatoric problems (such as guessing correct weight factors) associated with the so-called Feynman rules and the old-fashioned Wick's theorem, and the analysis does not even involve creation and annihilation operators. Finally, it avoids dealing with mass shell limits as in the LSZ formalism

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(cf. Weinberg, 1970), which becomes quite involved when one is dealing with higher spin fields. The basic idea of the formalism is to use the quantum action principle (Schwinger, 1951, 1954; Lam, 1965; Manoukian, 1985) or its modification (Manoukian, 1985) as applied to non-Abelian gauge theories (Manoukian, 1986a) followed by a systematic analysis of the concept of *stimulated emissions* by external sources developed earlier (Manoukian, 1986b). In Section 2, the expression for *transition* amplitudes is derived. Section 3 gives some applications to particles with different spins, and a comparison with the more standard approach is finally made. In the concluding section (Section 4) some additional remarks are made.

2. FUNCTIONALS AND TRANSITION AMPLITUDES

We consider the field operator $\phi_i(x)$ appearing in the Lagrangian density $L(\phi_i)$ to be coupled linearly to external c-number sources $K_i(x)$. (The Lagrangian density, of course, depends also on the derivatives of the fields.) A quantum mechanical parameter(s) will be denoted by λ. In general, we have the functional relation $\delta K_i(x)/\delta K_j(x')=\delta_{ij}\delta(x-x')$. Scattering in and out states are denoted by $|g_-\rangle$ and $|f_+\rangle$, respectively.

The quantum action (dynamical) principle (Schwinger, 1951, 1954; Lam, 1965; Manoukian, 1985) then reads

$$\frac{\partial}{\partial\lambda}\langle f_+|g_-\rangle^{K_i,\lambda}=i\int(dx)\,\frac{\partial}{\partial\lambda}L'\left(\frac{-i\delta}{\delta K_i}\right)\langle f_+|g_-\rangle^{K_i,\lambda}\tag{1}$$

which integrates to

$$\langle f_+|g_-\rangle\equiv\langle f_+|g_-\rangle^{0,\lambda}=\exp\left[i\int(dx)\,L_{\mathrm{I}}'\left(\frac{-i\delta}{\delta K_i}\right)\right]\langle f_+|g_-\rangle^{K_i,0}\bigg|_{K_i=0}\tag{2}$$

Here $(dx)=dx^0\,dx^1\,dx^2\,dx^3$; $L_{\mathrm{I}}'(-i\delta/\delta K_i)$, in general, denotes the interaction Lagrangian density with the fields ϕ_i in it replaced by $-i\delta/\delta K_i$. In gauge theories the interaction Lagrangian is supplemented (Manoukian, 1986a) by a Faddeev-Popov factor (Faddeev and Popov, 1967); see Section 3. The expression for $\langle f_+|g_-\rangle$ denotes the transition amplitude we are seeking. $\langle f_+|g_-\rangle^{K_i,0}$ denotes the transition amplitude in the presence of the external sources (Manoukian, 1986b) with no coupling ($\lambda\equiv 0$) between the particles.

For simplicity of the notation, consider a field theory of one type of real scalar particle with an interaction Lagrangian density $L_{\mathrm{I}}(\phi)$. Let $K(x)$ denote an external c-number source. Let

$$K(p)=\int(dx)\,e^{-ixp}K(x)\tag{3}$$

and in a convenient discretization notation (Schwinger, 1969, 1970; Manoukian, 1984) for the momentum variable we set

$$K_{\mathbf{p}} = (d\omega_{\mathbf{p}})^{1/2} K(p), \qquad d\omega_{\mathbf{p}} = d^3\mathbf{p}/2p^0, \qquad p^0 = +(\mathbf{p}^2 + m^2)^{1/2} \tag{4}$$

Consider the *scattering* of M particles, $M_{\mathbf{p}_1}$ of which each has a momentum $\mathbf{p}_1$; $M_{\mathbf{p}_2}$ of which each has a momentum $\mathbf{p}_2$; and so on ($M_{\mathbf{p}_1} + M_{\mathbf{p}_2} + \cdots = M$); *to* N particles, $N_{\mathbf{p}_1}$ of which each has a momentum $\mathbf{p}_1$; $N_{\mathbf{p}_2}$ of which each has a momentum $\mathbf{p}_2$; and so on ($N_{\mathbf{p}_1} + N_{\mathbf{p}_2} + \cdots = N$). Then (Manoukian, 1986b) $\langle f_+ | g_- \rangle^{K,0}$ is given by

$$\begin{aligned} &\langle N; N_{\mathbf{p}_1}, N_{\mathbf{p}_2}, \ldots | M; M_{\mathbf{p}_1}, M_{\mathbf{p}_2}, \ldots \rangle^K \\ &= (N_{\mathbf{p}_1}!\, N_{\mathbf{p}_2}! \ldots M_{\mathbf{p}_1}!\, M_{\mathbf{p}_2}! \ldots)^{1/2} \\ &\times \sum{}^* \frac{(iK_{\mathbf{p}_1})^{N_{\mathbf{p}_1} - m_{\mathbf{p}_1}}}{(N_{\mathbf{p}_1} - m_{\mathbf{p}_1})!} \frac{(iK_{\mathbf{p}_2})^{N_{\mathbf{p}_2} - m_{\mathbf{p}_2}}}{(N_{\mathbf{p}_2} - m_{\mathbf{p}_2})!} \cdots \frac{\langle 0_+ | 0_- \rangle^K}{m_{\mathbf{p}_1}!\, m_{\mathbf{p}_2}! \ldots} \cdots \\ &\times \frac{(iK^*_{\mathbf{p}_1})^{M_{\mathbf{p}_1} - m_{\mathbf{p}_1}}}{(M_{\mathbf{p}_1} - m_{\mathbf{p}_1})!} \frac{(iK^*_{\mathbf{p}_2})^{M_{\mathbf{p}_2} - m_{\mathbf{p}_2}}}{(M_{\mathbf{p}_2} - m_{\mathbf{p}_2})!} \cdots \end{aligned} \tag{5}$$

where $\sum^*$ stands for a summation over all nonnegative integers $m_{\mathbf{p}_1}, m_{\mathbf{p}_2}, \ldots$ *such that* $0 \le m_{\mathbf{p}_i} \le \min(N_{\mathbf{p}_i}, M_{\mathbf{p}_i})$, $i = 1, 2, \ldots$.

Here $\langle 0_+ | 0_- \rangle^K$ denotes the vacuum-to-vacuum transition amplitude, in the presence of the external source $K(x)$, and is given by the well-known expression

$$\langle 0_+ | 0_- \rangle^K = \exp\left[\frac{i}{2} \int (dx)\,(dx')\, K(x) \Delta_+(x - x') K(x') \right] \tag{6}$$

$$\Delta_+(x - x') = \int \frac{(dp)}{(2\pi)^4} \frac{e^{ip(x-x')}}{p^2 + m^2 - i\varepsilon}, \qquad \varepsilon \to +0 \tag{7}$$

As an example, consider the scattering of r particles with nonoverlapping momenta $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_r$ to s particles with nonoverlapping momenta $\mathbf{p}_{r+1}, \ldots, \mathbf{p}_{r+s}$, then equation (5) reads

$$\langle s; 1_{\mathbf{p}_{r+1}}, \ldots, 1_{\mathbf{p}_{r+s}} | r; 1_{\mathbf{p}_1}, \ldots, 1_{\mathbf{p}_r} \rangle^K = (i)^{r+s} K_{\mathbf{p}_1} \cdots K_{\mathbf{p}_r} K^*_{\mathbf{p}_{r+1}} \cdots K^*_{\mathbf{p}_{r+s}} \tag{8}$$

where $p_i^0 = (\mathbf{p}_i^2 + m^2)^{1/2}$, $i = 1, \ldots, r + s$. Hence the exact expression for the transition amplitude (2) is given from (8), (3), and (4) to be explicitly

$$\begin{aligned} &(i)^{l+s} \left[\prod_{j=1}^{r+s} \left(\frac{d^3\mathbf{p}_j}{2p_j^0} \right)^{1/2} \int dx_j \exp(-i\varepsilon_j x_j p_j) \right] \\ &\times \exp\left[i \int (dx)\, L_I'\left(\frac{-i\delta}{\delta K} \right) \right] \\ &\times K(x_1) \cdots K(x_r) K^*(x_{r+1}) \cdots K^*(x_{r+s}) \langle 0_+ | 0_- \rangle^K \end{aligned} \tag{9}$$

where $\varepsilon_j = +1$, for $j = 1, \ldots, r$; $\varepsilon_j = -1$ for $j = r+1, \ldots, r+s$; with the elementary rule that $\delta K(x)/\delta K(x') = \delta(x - x')$. The general expression corresponding to (5) has been worked out (Manoukian, 1986b) for higher spins as well, and a corresponding expression to the transition amplitude in (9) can then be written out.

3. APPLICATIONS AND COMPARISON WITH THE STANDARD APPROACH

For non-Abelian gauge theories with the vector field A^a_μ interacting with matter, one may take (Manoukian, 1986a), in the presence of external sources,

$$L = -\frac{1}{4} G^a_{\mu\nu} G^{\mu\nu a} + \frac{1}{2}\left[\left(\frac{\partial_\mu}{i}\bar{\psi}\right)\gamma^\mu\psi - \bar{\psi}\gamma^\mu \frac{\partial_\mu \psi}{i}\right] - m_0\bar{\psi}\psi$$
$$+ g_0\bar{\psi}\gamma_\mu A^{\mu a} t^a \psi + \bar{\eta}\psi + \bar{\psi}\eta + J^a_\mu A^{\mu a} \tag{10}$$

where

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_0 f^{abc} A^b_\mu A^c_\nu \tag{11}$$

in a standard notation. By working in the Coulomb gauge $\partial_k A^{ka} = 0$, $k = 1, 2, 3$, then the interaction Lagrangian density L'_I, not involving external sources, is automatically supplemented by the presence of a Faddeev–Popov factor (Manoukian, 1986a) and the expression for the transition amplitude becomes

$$\langle f_+ | g_- \rangle = \exp\left[i \int (dx)\; L'_I(A'^a_\mu, \psi', \bar{\psi}') \right.$$
$$\left. + \operatorname{Tr}\ln\left(\delta^{ab} + g_0 f^{acb} \frac{1}{\boldsymbol{\partial}^2} A'^c_k \partial^k\right)\right] \langle f_+ | g_- \rangle^{\eta, \bar{\eta}, J} \tag{12}$$

where $A'^a_\mu = -i\delta/\delta J^a_\mu$, $\psi' = -i\delta/\delta\bar{\eta}$, $\bar{\psi}' = -i\delta/\delta n$. The general expression $\langle f_+ | g_- \rangle^{\eta, \bar{\eta}, J}$ for free spin-one particles interacting with the external source J^a_μ, and free spin $\frac{1}{2}$ particles interacting with external sources η and $\bar{\eta}$, can be read directly from our earlier work (Manoukian, 1986b) and will not be repeated here.

Finally, we compare our expression for the transition amplitude with the standard approach. To this end we consider, for simplicity of notation only, a spin-$\frac{1}{2}$ field ψ interacting with a scalar field ϕ. The fields ψ, $\bar{\psi}$, and ϕ are coupled linearly to external sources $\bar{\eta}$, η, and K, respectively. The interaction Lagrangian density is taken to be $g_0\bar{\psi}\psi\phi$.

Upon defining

$$\left(2m\frac{d^3\mathbf{p}}{2p^0}\right)^{1/2}\bar{\eta}(p)u(p,\sigma)=\eta^*_{\mathbf{p}\sigma_-} \tag{13}$$

$$\left(2m\frac{d^3\mathbf{p}}{2p^0}\right)^{1/2}\bar{u}(p,\sigma)\eta(p)=\eta_{\mathbf{p}\sigma_-} \tag{14}$$

we can use equation (54) in Manoukian (1986b) and equation (5) above together with equation (2) to write the explicit expression for the transition amplitude. For simplicity consider the scattering process $p\phi\to p\phi$, where p is the spin-$\frac{1}{2}$ particle. Let the incoming and outgoing momenta and spins of the fermion be $(\mathbf{k}_1,\sigma_1)\neq(\mathbf{k}_2,\sigma_2)$, respectively. Let the incoming and outgoing momenta of the scalar particle be $\mathbf{q}_1\neq\mathbf{q}_2$. Then from (2), (5), (13) and (14) above, and equation (54) in Manoukian (1986b), we can write for the transition amplitude associated with the process $[(\mathbf{k}_1,\sigma_1),\mathbf{q}_1]\to[(\mathbf{k}_2,\sigma_2),\mathbf{q}_2]$:

$$\begin{aligned}
&(i)^4\left(2m\frac{d^3\mathbf{k}_1}{2k_1^0}\right)^{1/2}\left(2m\frac{d^3\mathbf{k}_2}{2k_2^0}\right)^{1/2}\left(\frac{d^3\mathbf{q}_1}{2q_1^0}\right)^{1/2}\left(\frac{d^3\mathbf{q}_2}{2q_2^0}\right)^{1/2}\\
&\times\int(dx_1)\cdots(dx_4)\exp(-ik_1x_1)\\
&\times\exp(ik_2x_2)\exp(-iq_1x_3)\exp(iq_2x_4)\\
&\times\bar{u}(k_1,\sigma_1)\exp\left[ig_0\int(dx)\,(-i)\frac{\delta}{\delta\eta(x)}(-i)\frac{\delta}{\delta\bar{\eta}(x)}(-i)\frac{\delta}{\delta K(x)}\right]\\
&\times\eta(x_1)\bar{\eta}(x_2)K(x_3)K^*(x_4)u(k_2,\sigma)\\
&\times\langle 0_+|0_-\rangle^K\langle 0_+|0_-\rangle^{\eta,\bar{\eta}}\big|_{K=0;\eta,\bar{\eta}=0}
\end{aligned} \tag{15}$$

where $\langle 0_+|0_-\rangle^K$ is given in (6) and

$$\langle 0_+|0_-\rangle^{\eta,\bar{\eta}}=\exp\left[i\int(dx)\,(dx')\,\bar{\eta}(x)S_+(x-x')\eta(x')\right] \tag{16}$$

$$S_+(x-x')=\int\frac{(dp)}{(2\pi)^4}\frac{\{\exp[ip(x-x')]\}(-\gamma p+m)}{p^2+m^2-i\varepsilon},\qquad \varepsilon\to+0 \tag{17}$$

Now for any functional $F[K]$ of K we may use the property of the translation operator

$$\exp\left[\int(dx)\,f(x)\frac{\delta}{\delta K(x)}\right]F[K]=F[K+f] \tag{18}$$

to write

$$\exp\left[i\int (dx)\, L_I'\right]\eta(x_1)\bar{\eta}(x_i)K(x_3)K(x_4)\langle 0_+|0_-\rangle^K\langle 0_+|0_-\rangle^{\eta,\bar{\eta}}$$

$$=\left[\eta(x_1)-g_0\frac{\delta}{\delta K(x_1)}\frac{\delta}{\delta\bar{\eta}(x_1)}\right]$$

$$\times\left[\bar{\eta}(x_2)-g_0\frac{\delta}{\delta K(x_2)}\frac{\delta}{\delta\eta(x_2)}\right]$$

$$\times\left[K(x_3)-g_0\frac{\delta}{\delta\eta(x_3)}\frac{\delta}{\delta\bar{\eta}(x_3)}\right]$$

$$\times\left[K(x_4)-g_0\frac{\delta}{\delta\eta(x_4)}\frac{\delta}{\delta\bar{\eta}(x_4)}\right]\langle 0_+|0_-\rangle^{K,\eta,\bar{\eta}}_{\text{exact}} \tag{19}$$

where

$$\langle 0_+|0_-\rangle^{K,\eta,\bar{\eta}}_{\text{exact}}\equiv\langle 0_+|0_-\rangle$$

is (up to an unimportant numerical factor) the transition amplitude in the presence of the external sources, for $g_0\neq 0$.

The functional differential equations are

$$\left(\frac{\gamma\partial}{i}+m_0\right)(-i)\frac{\delta}{\delta\bar{\eta}(x)}\langle 0_+|0_-\rangle$$

$$=\left[\eta(x)+g_0(-i)\frac{\delta}{\delta K(x)}(-i)\frac{\delta}{\delta\bar{\eta}(x)}\right]\langle 0_+|0_-\rangle \tag{20}$$

$$\left(\frac{\gamma\partial}{i}-m_0\right)(-i)\frac{\delta}{\delta\eta(x)}\langle 0_+|0_-\rangle$$

$$=-\left[\bar{\eta}(x)+g_0(-i)\frac{\delta}{\delta K(x)}(-i)\frac{\delta}{\delta\eta(x)}\right]\langle 0_+|0_-\rangle \tag{21}$$

$$(-\Box+M_0^2)(-i)\frac{\delta}{\delta K(x)}\langle 0_+|0_-\rangle$$

$$=\left[K(x)+g_0(-i)\frac{\delta}{\delta\eta(x)}(-i)\frac{\delta}{\delta\bar{\eta}(x)}\right]\langle 0_+|0_-\rangle \tag{22}$$

From (15) and (19)-(22) we obtain the familiar LSZ expression for the transition amplitude:

$$
\begin{aligned}
&(i)^4\left(2m\frac{d^3\mathbf{k}_1}{2k_1^0}\right)^{1/2}\left(2m\frac{d^3\mathbf{k}_2}{2k_2^0}\right)^{1/2}\left(\frac{d^3\mathbf{q}_1}{2q_1^0}\right)^{1/2}\left(\frac{d^3\mathbf{q}_1}{2q_2^0}\right)^{1/2}\\
&\times\int (dx_1)\cdots(dx_4)\, e^{-ik_1x_1}\, e^{ik_2x_2}\, e^{-q_1x_3}\, e^{iq_2x_4}\\
&\times\bar{u}(k_1,\sigma_1)\left[\left(\frac{\gamma\partial^1}{i}+m_0\right)\left(\frac{\gamma\partial^2}{i}-m_0\right)(-\Box_3^2+M_0^2)\right.\\
&\left.\times(-\Box_4^2+M_0^2)\tau(x_1,x_2,x_3,x_4)\right]u(k_2,\sigma_2)
\end{aligned}
\tag{23}
$$

where

$$
\begin{aligned}
\tau(x_1,x_2,x_3,x_4)=(-i)\frac{\delta}{\delta\bar{\eta}(x_1)}(-i)\frac{\delta}{\delta\eta(x_2)}(-i)\\
\times\frac{\delta}{\delta K(x_3)}(-i)\frac{\delta}{\delta K(x_4)}\langle 0_+|0_-\rangle\bigg|_{K=0,\eta=0,\bar{\eta}=0}
\end{aligned}
\tag{24}
$$

This completes the demonstration of the consistency of our formalism with the standard approach.

4. CONCLUSIONS

The functional expression for transition amplitudes [e.g., equation (9)] has several advantages over the standard approaches. Many of these were mentioned in the introduction. In particular, we note also that since our formalism does not involve the full Green's function, it is quite suitable for systematic perturbative analyses, and for the problem of renormalization, in general. The gauge problem of *transition* amplitudes may be also discussed directly within our formalism from the known gauge properties of the vacuum-to-vacuum transition amplitudes studied in detail for Abelian and non-Abelian gauge theories in Manoukian (1986a). Such an analysis of the gauge problem of transition amplitudes themselves will be carried out in a future report.

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